Electromagnetic theory

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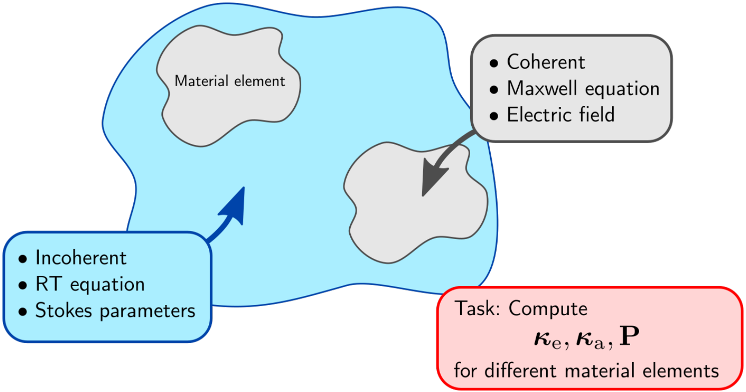
Outline

Outline

The problem at a glance

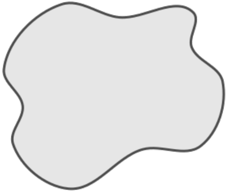
SMRT's main task: Solve RTE

$$\mu \frac{\partial \mathbf{I}(\mu,\phi,z)}{\partial z} = -\boldsymbol{\kappa}_{\mathrm{e}}(\mu,\phi,z)\,\mathbf{I}(\mu,\phi,z) + \frac{1}{4\pi}\iint_{4\pi} \mathbf{P}(\mu,\phi;\mu',\phi',z)\mathbf{I}(\mu',\phi',z)\,d\Omega' + \boldsymbol{\kappa}_{\mathrm{a}}(\mu,\phi,z)\,\alpha T(z)\mathbf{1}$$

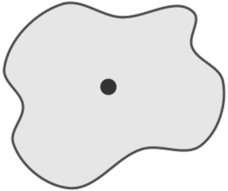


The material elements we have to deal with:

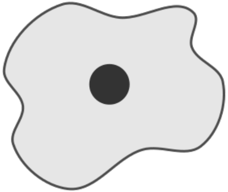
Homogeneous:



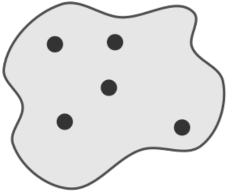
Small sphere:



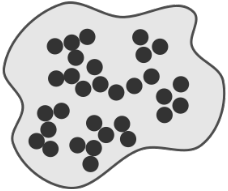
Large sphere:



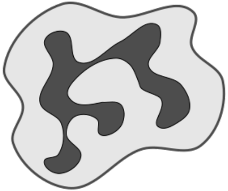
Sparse spheres:



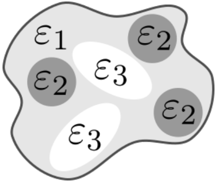
Dense spheres



Arbitrary bicontinuous:



Always the same task:



Given a random material: ε_1 (air), ε_2 (ice), ε_3 (brine)

Solve the Maxwell equation for the electric field $\boldsymbol{E}$ inside the material element:

$$\nabla \times \nabla \boldsymbol{E}(\boldsymbol{r}) - \frac{k_0^2}{\varepsilon_0} \varepsilon(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r}) = 0 \tag{1}$$

with vacuum wave number $k_0 = 2\pi\nu / c_0$, frequency ν, speed of light c_0 and position dependent permittivity

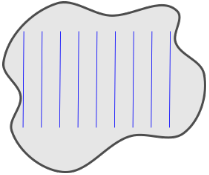
$$\varepsilon(\boldsymbol{r}) = \begin{cases} \varepsilon_1 \text{ if } \boldsymbol{r} \text{ is in air} \\ \varepsilon_2 \text{ if } \boldsymbol{r} \text{ is in ice} \\ \varepsilon_3 \text{ if } \boldsymbol{r} \text{ is in brine} \end{cases} .$$

Apparently managable, but nasty at heart...

Outline

Recap I: Terminology, plane waves

The homogeneous case: If ε in (1) does not depend on position:



Plane wave solution:

$$\boldsymbol{E}(\boldsymbol{r}, t) \quad = \quad \boldsymbol{E_0} \exp(ik\widehat{\boldsymbol{k}} \cdot \boldsymbol{r} - i\omega t)$$

with a complex *propagation constant* k

$$k = k' + ik''$$

which is related to the complex *index of refraction* n

$$k \quad = \quad nk_0$$

which is in turn related to the complex *dielectric constant* ε (or *dielectric permittivity*)

$$\varepsilon = n^2$$

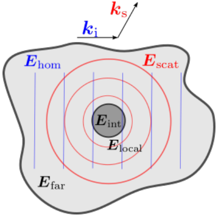
All quantities k, n, ε are equivalent, complex-valued, material properties (interchangably used in literature) of the homogeneous medium.

Recap II: Single sphere scattering

Perfect homogeneity is an idealization valid only for $\omega \to 0$ (low freq / static). By increasing the frequency, the plane wave will start to "see" heterogeneities (always existing) $\to$ scattering.

Scattering at a dielectric sphere: Decomposition

$$\boldsymbol{E} = \boldsymbol{E}_{hom} + \boldsymbol{E}_0 f(\boldsymbol{k}_s, \boldsymbol{k}_i) \frac{\exp ikr}{r} \tag{2}$$



In this case:

- Distinction between hom background scat
- Distinction between far, mean, local and internal field

Nature of exact solutions are controlled by size (a/λ):

- $a/\lambda \ll 1$: Rayleigh, $a/\lambda \approx 1$: Mie, $a/\lambda \gg 1$: Geometrical optics

Recap III: Single sphere RTE properties

For a single sphere everything follows directly from the exact (Rayleigh) solution, e.g:

Phase function: (units m^2)

$$p(\chi) \sim a^6 k^4 \sin^2 \chi \left| \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + 2\varepsilon_1} \right|^2$$

(intensity distribution of a dipole, scattering angle χ)

Scattering coefficient: (units m^2)

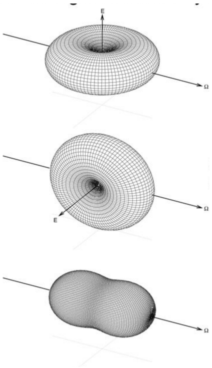
$$\kappa_s \sim k^4 a^6 \left| \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + 2\varepsilon_1} \right|^2$$

(scattered intensity integrated over all directions).

Absorption coefficient: (units m^2)

$$\kappa_a \sim k a^3 \frac{\varepsilon_2''}{\varepsilon_1} \left| \frac{3\varepsilon_1}{\varepsilon_2 + 2\varepsilon_1} \right|^2$$

(Ohmic dissipation from the internal field integrated over the sphere)



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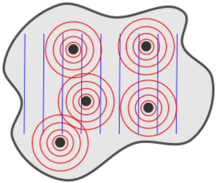
Recap IV: RTE properties and general definitions from coherent waves

Scattering coefficient:	$\kappa_s = \frac{1}{4\pi}\int_{4\pi} d\Omega \vert f(\boldsymbol{k}_s, \boldsymbol{k}_i)\vert^2$
	Total scattered intensity
Phase function	$p = \frac{4\pi}{\kappa_s}\vert f(\boldsymbol{k}_s, \boldsymbol{k}_i)\vert^2$
	Angular distribution of scattered intensity
Extinction coefficient:	$\kappa_e = \frac{4\pi}{k}\mathcal{I}m\big(f(\boldsymbol{k}_i, \boldsymbol{k}_i)\big)$, $\kappa_e = 2\mathcal{I}m(k_{eff})$
	Intensity attenuation (via optical theorem or the effective propagation constant)
Absorption coefficient:	$\kappa_a = \kappa_e - \kappa_s$, $\kappa_a = \frac{\omega}{2}\int_V dV \varepsilon''_{int}\vert E_{int}\vert^2$
	Intensity attenuation due to Ohmic currents
Dielectric permittivity:	"$\varepsilon_{\text{eff}} = k_{eff}^2 / k_0^2$"
	Static polarizability ($\lim_{\omega\to 0}$ is often implied)

Different roads leading to Rome $\rightarrow$ unification of different results is tedious.

How approximations for random media are constructed:

Maxwell Eq (1) for $\boldsymbol{E}$ is commonly rewritten



$$\nabla \times \nabla \boldsymbol{E}(\boldsymbol{r}) - \frac{k_0}{\varepsilon_0}\varepsilon_{hom}\boldsymbol{E}(\boldsymbol{r}) = \frac{k_0}{\varepsilon_0}[\varepsilon(\boldsymbol{r}) - \varepsilon_{hom}]\boldsymbol{E}(\boldsymbol{r})$$

as a perturbation scheme around a homogeneous background ε_{hom} and fluctuations $[\varepsilon(\boldsymbol{r}) - \varepsilon_{hom}]$ as scattering sources.

For complex media:

- the choice of hom is rather arbitrary
- everything relies on proper approximtions for mean, local and internal fields.

Examples:

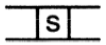
- Take hom as air (QCA, SFT)
- Compute hom self-consistently (QCA-CP)
- Take hom as "apparent medium" (IBA)
- "Good" choices for hom depend on the microstructure (Rechtsman 2008)

Summary EM for microwave modeling: Everything is around...

... you just have to decrypt it:

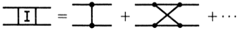
$$
\begin{aligned}
\langle \overline{\overline{G}}(\bar r,\bar r_o)\overline{\overline{G}}^*(\bar r',\bar r'_o)\rangle &= \overline{\overline{G}}^{(0)}(\bar r,\bar r_o)\overline{\overline{G}}^{(0)*}(\bar r',\bar r'_o)\\
&+\overline{\overline{G}}^{(0)}(\bar r,\bar r_o)\int d\bar r'_1\int d\bar r'_2\overline{\overline{G}}^{(0)*}(\bar r',\bar r'_1)\cdot\overline{\overline{G}}^{(0)*}(\bar r'_1,\bar r'_2)\cdot\overline{\overline{G}}^{(0)*}(\bar r'_2,\bar r'_o)\\
&\quad\langle Q^*(\bar r'_1)Q^*(\bar r'_2)\rangle\\
&+\int d\bar r_1\int d\bar r_2\overline{\overline{G}}^{(0)}(\bar r,\bar r_1)\cdot\overline{\overline{G}}^{(0)}(\bar r_1,\bar r_2)\cdot\overline{\overline{G}}^{(0)}(\bar r_2,\bar r_o)\langle Q(\bar r_1)Q(\bar r_2)\rangle\overline{\overline{G}}^{(0)*}(\bar r',\bar r'_o)\\
&+\overline{\overline{G}}^{(0)}(\bar r,\bar r_o)\int d\bar r'_1\int d\bar r'_2\int d\bar r'_3\overline{\overline{G}}^{(0)*}(\bar r',\bar r'_1)\cdot\overline{\overline{G}}^{(0)*}(\bar r'_1,\bar r'_2)\cdot\overline{\overline{G}}^{(0)*}(\bar r'_2,\bar r'_3)\cdot\\
&\quad\overline{\overline{G}}^{(0)*}(\bar r'_3,\bar r'_o)\langle Q^*(\bar r'_1)Q^*(\bar r'_2)Q^*(\bar r'_3)\rangle\\
&+\int d\bar r_1\int d\bar r_2\int d\bar r_3\overline{\overline{G}}^{(0)}(\bar r,\bar r_1)\cdot\overline{\overline{G}}^{(0)}(\bar r_1,\bar r_2)\cdot\overline{\overline{G}}^{(0)}(\bar r_2,\bar r_3)\cdot\overline{\overline{G}}^{(0)}(\bar r_3,\bar r_o)\\
&\quad\langle Q(\bar r_1)Q(\bar r_2)Q(\bar r_3)\rangle\overline{\overline{G}}^{(0)*}(\bar r',\bar r'_o)\\
&+\int d\bar r_1\int d\bar r'_1\int d\bar r'_2\overline{\overline{G}}^{(0)}(\bar r,\bar r_1)\cdot\overline{\overline{G}}^{(0)}(\bar r_1,\bar r_o)\overline{\overline{G}}^{(0)*}(\bar r',\bar r'_1)\cdot\overline{\overline{G}}^{(0)*}(\bar r'_1,\bar r'_2)\cdot\\
&\quad\overline{\overline{G}}^{(0)*}(\bar r'_2,\bar r'_o)\langle Q(\bar r_1)Q^*(\bar r'_1)Q^*(\bar r'_2)\rangle\\
&+\int d\bar r_1\int d\bar r_2\int d\bar r'_1\overline{\overline{G}}^{(0)}(\bar r,\bar r_1)\cdot\overline{\overline{G}}^{(0)}(\bar r_1,\bar r_2)\cdot\overline{\overline{G}}^{(0)}(\bar r_2,\bar r_o)\overline{\overline{G}}^{(0)*}(\bar r',\bar r'_1)\cdot\\
&\quad\overline{\overline{G}}^{(0)*}(\bar r'_1,\bar r'_o)\langle Q(\bar r_1)Q(\bar r_2)Q^*(\bar r'_1)\rangle+\cdots
\end{aligned}
\tag{4.2.4}
$$

Feynman diagrams are introduced in conjunction with those of the previous section:

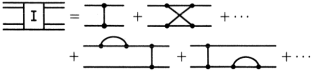


$$\equiv\langle\overline{\overline{G}}(\bar r,\bar r_o)\overline{\overline{G}}^*(\bar r',\bar r'_o)\rangle \tag{4.2.5}$$

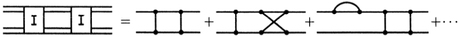
The sum of all strongly connected diagrams in (4.2.13) may be written in terms of the intensity operator as



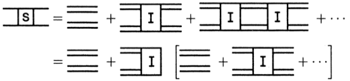
The sum of all strongly and weakly connected diagrams in (4.2.13) containing one intensity operator is given by



The sum of all weakly connected diagrams in (4.2.13) containing two intensity operators in cascade is given by

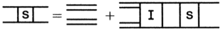


Continuing with this process, the series for the field correlation can be rewritten in the form



(4.2.16)

The key point is that weakly connected diagrams can be reproduced from strongly connected diagrams on iteration. On summation of (4.2.16), we have



(4.2.17)

(Pathways for the solution of Eq (1), taken from *Scattering of electromagnetic Waves* Pt3, Ch 4)

Outline

Introduction

EM theory in a nutshell

EM ingredients in SMRT

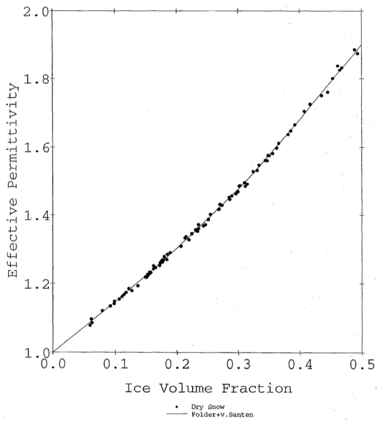
Effective permittivity: Dielectric mixing formulas

Polder–van Santen (SMRT default)

- self-consistent, effective-medium
- follows (Bruggeman 1935)
- "randomly oriented spheroids"

Maxwell–Garnett (option)

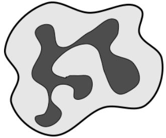
- not self-consistent,
- treats ice grains in air



(Mätzler 1996)

The improved Born approximation (IBA)

In IBA the perturbative solution of (1) is considers random two-phase microstructures:



- Admissible microstructures in SMRT: All

IBA scattering amplitude:

$$f_{scat} = f_2(1-f_2)M(|\mathbf{k}_d|)k_0^4 \ \sin^2\chi(\varepsilon_2-\varepsilon_1)^2 \ Y^2(\varepsilon_1,\varepsilon_2)$$

where

$f_2(1-f_2)M(\lvert\mathbf{k}_d\rvert)$	Microstructure term (FT of the autocorrelation function)
$k_0^4 \ \sin^2\chi$	Scattering geometry
$(\varepsilon_2-\varepsilon_1)^2$	Dielectric contrast
$Y^2(\varepsilon_1,\varepsilon_2)$	Improvement term for the internal field

($\rightarrow$ compare color-by-color to single sphere Rayleigh scattering)

Phase matrix:

RTE in SMRT considers all Stokes components

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} |\boldsymbol{E}_\mathrm{H}|^2 + |\boldsymbol{E}_\mathrm{V}|^2 \\ |\boldsymbol{E}_\mathrm{H}|^2 - |\boldsymbol{E}_\mathrm{V}|^2 \\ 2\mathcal{R}e(\boldsymbol{E}_\mathrm{H}\boldsymbol{E}_\mathrm{V}*) \\ 2\mathcal{R}e(\boldsymbol{E}_\mathrm{V}\boldsymbol{E}_\mathrm{H}*) \end{bmatrix}$$

IBA phase matrix

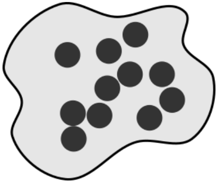
$$\mathbf{P}(\mu, \phi, \mu', \phi') = \begin{bmatrix} P_{11} & P_{12} & P_{13} & 0 \\ P_{21} & P_{22} & P_{23} & 0 \\ P_{31} & P_{32} & P_{33} & 0 \\ 0 & 0 & 0 & P_{44} \end{bmatrix}$$

can be computed from f_{scat} in the 1-2 frame (details in Picard et al 2018)

- Rayleigh form comes from “isotropic” and “low frequency” used in IBA

Alternative to IBA: The DMRT variants QCA vs QCA-CP

In DMRT (Dense media radiative transfer) the perturbative solution of (1) is constructed from sphere scattering properties (2):



- Admissible microstructures: only sphere models
- Mainly *sticky hard spheres* ($\rightarrow$ microstructure lecture)

Lingo:

- QCA: Quasi-crystalline approximation
- QCA-CP: Quasi-crystalline approximation with coherent potential

Mostly tackled by Tsang's group to get expressions for the effective propagation constant k_{eff}

Comparison of QCA vs QCA-CP

QCA-CP:

$$k_{eff}^2 = k_1^2 + n \frac{v_a z}{1 + z(1 - f_2)/(3k_{eff}^2)} \left\{ 1 + i \frac{2}{9} k_{eff} a^3 \frac{z}{1 + z(1 - f_2)/(3k_{eff}^2)} S(0) \right\}$$

($z = (k_2^2 - k_1^2)$, $n = N/V$, v_a sphere volume, cf. Eq. 5.3.124 Tsang III)

QCA:

$$k_{eff}^2 = k_1^2 + n \frac{3 v_a k_1^2 y}{1 - f_2 y} \left\{ 1 + i \frac{2}{3} \frac{(k_1 a)^3 y}{1 - fy} S(0) \right\}$$

($y = (\varepsilon_2 - \varepsilon_1)/(\varepsilon_2 + 2\varepsilon_1)$, $n = N/V$, v_a sphere volume, cf. Eq. 5.3.114 Tsang III)

- **Difference**: QCA-CP (self consistent background, k_{eff} also on the RHS) in contrast to QCA (air background only k_1 on the RHS)
- **Similarity**: Low frequency expansion, involving the structure factor $S(0)$ of the sphere packing (related to $\widetilde{C}(0)$ $\rightarrow$ microstructure lecture).

Comparison of IBA and QCA-CP

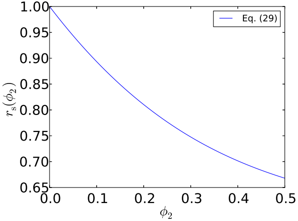
Scattering coefficient κ_s:

$$\kappa_s^{\mathrm{IBA}} = \frac{2}{9} k_0^4 a^3 \phi_2 f^{\mathrm{IBA}}(\varepsilon_1, \varepsilon_2, \phi_2)\, \widetilde{C}(0)$$

$$\kappa_s^{\mathrm{QCA-CP}} = \frac{2}{9} k_0^4 a^3 \phi_2 f^{\mathrm{QCA-CP}}(\varepsilon_1, \varepsilon_2, \phi_2)\, \widetilde{C}(0)$$

Main messages:

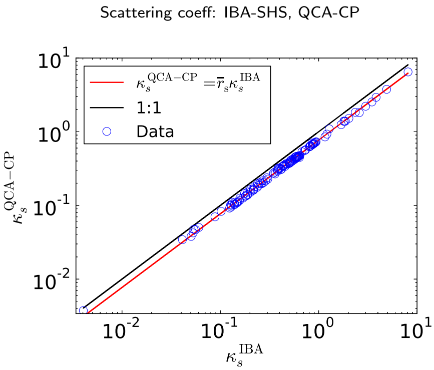
- "Slight difference in dielectrics": f^{IBA} vs $f^{\mathrm{QCA-CP}}$, ratio r_s:
- "No difference in the microstructure": $\widetilde{C}(0)$

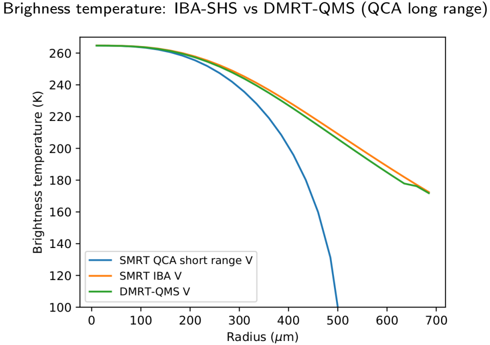


Relevant length scale hidden in:

- $\widetilde{C}(0)$: Fourier transform of the correlation function at the origin (units [m^3]!)

Comparison of IBA and QCA /QCA-CP





Main message: IBA (with SHS) and QCA/QCA-CP (long range) are very similar.

Summary: EM for SMRT

To understand the details of microwave modeling...

- ... some degree of EM is indispensible ☹️

SMRT...

- ... encapsulates EM whereever possible by cafefully chosen default behavior 🙂

However...

- ... we decided also not to use defaults on essential things, like microstructure...

Thank you for your attention.